Interval Estimation for Quantile on Two Parameters Exponential Distribution Under Multiple Type-II Censoring on Complex Case with Bootstrap Percentile

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ABSTRACT

In this article, two methods are proposed to give the interval estimation for quantile on two parameters exponential distribution under multiply type II censoring. The interval estimation for quantile can be constructed the estimated parameters. Those researchers have use approximate maximum likelihood estimator to construct interval estimation for two parameters exponential distribution under multiply type II censoring. All of these method need an assumption that sample is exponentially distributed. We will use another method, known as the bootstrap percentile. This method gives shorter interval than the traditional method and this method does not need an assumption that the sample is distributed exponentially.

Keywords: bootstrap percentile, multiply type II censoring, quantile

INTRODUCTION

The exponential distribution has immensely contributed in the analysis of lifetime. Historically the exponential distribution was the first lifetime model in which statistical methods in survival analysis were extensively developed. What distinguishes survival analysis from other fields of statistics is censoring. Vaguely speaking, a censored observation contains only partial information about the random variable of interest. The three types of censoring are the type-I censoring, the type-II censoring and random censoring (Kleinbaum & Klein 2005).

Balakrishnan, Kundu & Kannan (2007), Kang (2003), Balasubramanian & Balakrishnan (1992) and Balakrishnan (1990) use approximate maximum likelihood estimator to construct point estimation for one parameter exponential distribution under multiple type-II censoring on complex case. Fei & Kong (1994) use traditional method to construct interval estimation for one parameter exponential distribution under multiple type-II censoring on complex case. This interval needs an assumption that sample is exponentially distributed. Bootstrap method is a computer-based method for assigning measures of accuracy to statistical estimates, especially to calculate the confidence interval. The aim of using bootstrap method is to gain the best estimation from minimal data (Efron & Thibshirani 1993).

Fauzy *et al*. (2002a, 2002b, 2003a, 2003b, and 2005) used bootstrap method to construct interval estimation for quantile on one and two parameters exponential distribution under single type II censoring, double type II censoring and multiple type II censoring on simple case. In this paper comparison study will be made on the interval estimation for quantile on two parameters exponential distribution under multiply type II censoring on complex case using both the conventional method and bootstrap percentile method.

METHODS

The data used for illustration is taken from Balasubramanian & Balakrishnan (1992), Fei & Kong (1994) and Kang (2003). We begin with the interval estimation for quantile on two parameters exponential distribution under multiple type-II censoring on complex case using the traditional method. This will be followed with the searching of convergence condition from bootstrap's repeated samples. Next, after knowing the convergence condition, confidence interval for quantile on two parameters exponential distribution under multiple type-II censoring on complex case with bootstrap percentile method will be computed.

Two parameter exponential distribution

The two parameters exponential distribution has probability density function (Lawless 2003):

$$f(x;\eta,\theta)=\frac{1}{\theta}\exp\left(-\frac{x-\eta}{\theta}\right);x\geq\eta,\theta\geq 0 \quad (1)$$

Here η is the warranty time and θ is the residual mean life. Suppose n items are placed on a lifetesting experiment and that the r_1th, r_2th, $\cdots$, r_kth failure times ate only made available. That is,

$T_1 = X_{r_1:n}, T_2 = X_{r_2:n}, \cdots, T_k = X_{r_k:n}$ (2)

is the multiply type-II censored sample available from (1).

The likelihood function based on the multiply type-II censoring on complex case in (2) is given by:

$$L = \text{const.}\,\theta^{-k}\left(1-\exp\left(-\frac{(t_i-\eta)}{\theta}\right)\right)^{r_1-1}\exp\left(\frac{\eta}{\theta}(n-r_1+1)\right)\exp\left(-\frac{\sum_{i=1}^{k}t_i+(n-r_k)t_k}{r}\right)\times\prod_{i=1}^{k-1}\left(\exp\left(-\frac{t_i}{\theta}\right)-exp\left(-\frac{t_{i+1}}{\theta}\right)\right)^{r_{i+1}-r-1} \quad (3)$$

From (3) the likelihood equation for θ is obtained as:

$$\frac{\partial \ln L}{\partial\theta}=\frac{1}{\theta^2}\left[k\theta+(n-r_1+1)t_1-\left(\sum_{i=1}^{k}t_i+(n-r_k)t_k\right)-\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\left(\frac{-t_{i+1}\exp\left(-\frac{t_{i+1}}{\theta}\right)+t_i\exp\left(-\frac{t_i}{\theta}\right)}{-\exp\left(-\frac{t_{i+1}}{\theta}\right)+\exp\left(-\frac{t_i}{\theta}\right)}\right)\right] \quad (4)$$

The estimation for θ is (Balasubramanian and Balakrishnan, 1992):

$$\hat{\theta}=\frac{\sum_{i=1}^{k}t_i+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\ (\delta_i t_i+(1-\delta_i)t_{i+1})+(n-r_k)t_k-(n-r_1+1)t_1}{k-\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\gamma_i} \quad (5)$$

The (1- α) lower confidence bound for θ is (Fei and Kong, 1994):

$$\hat{\theta}_{\min}=\frac{4\mu_{k:1}\left[\sum_{i=2}^{k}t_{r_i:n}-(k-1)t_{r_1:n}\right]}{V_{k:1}\,\chi^2_{1-\alpha}\left(\frac{2\mu^2_{k:1}}{V_{k:1}}\right)}=\frac{4\mu_{k:1}\left[\sum_{i=1}^{k}t_{r_i:n}-k\,t_{r_1:n}\right]}{V_{k:1}\,\chi^2_{1-\alpha}\left(\frac{2\mu^2_{k:1}}{V_{k:1}}\right)} \quad (6)$$

where $\mu_{k:1}=2\sum_{i=2}^{k}\sum_{j=r_{i-1}+1}^{r_i}\frac{k+1-i}{n-j+1}$ and $V_{k:1}=4\sum_{i=2}^{k}\sum_{j=r_{i-1}+1}^{r_i}\frac{(k+1-i)^2}{(n-j+1)^2}$

For the case when $r_1 = 1$, we simply observe from (3) that L is a monotonic increasing function in η and hence the maximum likelihood estimator for η is simply $\hat{\eta}=t_1=x_{r_1:n}=x_{1:n}$..

For the case when $r_1 > 1$ the likelihood equation for η:

$$\frac{\partial \ln L}{\partial\eta}=\frac{1}{\theta}\left[(n-r_1+1)-(r_1-1)\frac{\text{eksp}\left(-\left(\frac{t_1-\eta}{\theta}\right)\right)}{1-\text{eksp}\left(-\left(\frac{t_1-\eta}{\theta}\right)\right)}\right]=0 \quad (7)$$

yields the maximum likelihood estimator for η to be:

$$\hat{\eta}=t_1+\hat{\theta}\ln\left(\frac{n-r_1+1}{n}\right) \quad (8)$$

The (1- α) lower confidence bound for η is (Fei and Kong, 1994):

$$\hat{\eta}_{\min}=t_{r_1:n}-\frac{\mu_1}{\mu_{k:1}}F_{1-\alpha}(f_1,f_2)\left[\sum_{i=1}^{k}t_{r_i:n}-k\,t_{r_1:n}\right] \quad (9)$$

where $f_1=\frac{2\mu_1^2}{V_1}$ and $f_2=\frac{2\mu_{k:1}^2}{V_{k:1}}$

The pth quantile on one parameter exponential distribution, given by tp, and so:

$$t_p=\hat{\mu}-\ln(1-p)\hat{\theta} \quad (10)$$

The (1-α) confidence for survivor function is:

$$\hat{\mu}_{\min}-\log(1-p)\hat{\theta}_{\min} \langle\ t_p\ \langle\ \hat{\mu}_{\max}-\log(1-p)\hat{\theta}_{\max} \quad (11)$$

Bootstrap percentile method

In setting up of the bootstrap method to find the confidence intervals and estimating significance levels, the method consists of approximating the distribution of a function of the observations and the underlying distribution, such as the pivot, denoted by Efron as the bootstrap distribution of this quantity. This distribution is obtained by replacing the unknown distribution by the empirical distribution of the data in the definition of the statistical function, and then resampling the data to obtain a Monte Carlo distribution for the resulting random variable (Chernick 1999).

Bootstrap method is a computer-based method for assigning measures of accuracy to statistical estimates, especially to calculate the confidence interval. Bootstrap itself comes from the phrase “pull oneself up by one’s Bootstrap” which means to stand up by one’s own feet and do with minimal resources. The minimal resource is a minimum data, data which are free from certain assumption or data with no assumption at all about the population distribution. The aim of using bootstrap method is to gain the best estimation from minimal observation (Chernick 2007).

The Bootstrap’s percentile procedures for the interval estimation for quantile on two parameters exponential distribution under multiple type-II censoring on complex case are as follows:

1. give an equal opportunity $\frac{1}{n}$ to every observation,
2. take *n* sample with replication,
3. do step 2 until *B* times in order to get an “*independent bootstrap replications*”, $\hat{\beta}^{*1},\hat{\beta}^{*2},\dots,\hat{\beta}^{*\bar{D}}$ and search for convergence condition. Calculate:

$$t_p^{*b}=\hat{\mu}^{*b}-\ln(1-p)\hat{\theta}^{*b}\ \text{with} \quad (12)$$

$$\hat{\eta}^{*b}=t_{1:n}^{*b}+\hat{\theta}^{*b}\,ln\left(\frac{n-r_1+1}{n}\right)$$

for $r>1$ or

$$\hat{\eta}^{*b} = t_1^{*b} = x_{r_1:n}^{*b} = x_{1:n}^{*b} \quad \text{for } r = 1$$

$$\hat{\theta}^{*b} = \frac{\sum_{i=1}^{k} t_i^{*b} + \sum_{i=1}^{k-1}(r_{i+1} - r_i - 1)\left(\delta_i t_i^{*b} + (1-\delta_i) t_{i+1}^{*b}\right) + (n - r_k) t_k^{*b} - (n - r_1 + 1) t_1^{*b}}{k - \sum_{i=1}^{k-1}(r_{i+1} - r_i - 1)\gamma_i}$$

4. define the confidence interval at the level (1-α) of the bootstrap percentile for quantile on two parameters exponential distribution under multiple type-II censoring on complex case as:

$$\left[t_p^{*B(\alpha/2)}, t_p^{*B(1-\alpha/2)} \right] \quad (13)$$

Expressions (13) refer to the ideal bootstrap situation where the bootstrap replications are infinite. So if $B = 2000$ and $\alpha = 0.05$, $\hat{\theta}^{*\bar{D}(\alpha/2)}$ is the $50th$ and $\hat{\theta}^{*\bar{D}(1-\alpha/2)}$ is the $1950th$ ordered value of the replications.

RESULTS AND DISCUSSION

We consider the data presented in Balasubramanian and Balakrishnan (1992), Fei and Kong (1994) and Kang (2003). It is a simulated lifetime data set (in hours) from two parameters exponential distribution with θ = 20.

The data is given as follow:

0.961	0.990	1.565	2.031	2.204
2.340	3.642	6.008	6.538	7.145
-	-	-	11.937	15.433
18.234	18.307	22.096	-	-
-	28.799	30.692	30.737	33.702
34.245	-	-	-	-

We will construct interval estimation for 0.30 and 0.70 quantile ($t_{0,30}$ and $t_{0,70}$). From equations (10) for traditional method and (12) for bootstrap methods, the point estimates of:

$t_{0,30}$= 7.680639 and $t_{0,30}^{*b}$ = 8.678485

$t_{0,70}$ = 23.643450 and $t_{0,70}^{*b}$ = 26.222990

Based on the above under multiple type-II censoring on complex case, the intervals estimation for 0.30 quantile in Table 1 and for 0.70 quantile in Table 2. Bootstrap's repeated result gives a convergence condition that begins at B = 4000 for 0.30 quantile and B = 4000 for 0.70 quantile. The plot between bias and replication are shown in Figure 1 for 0.30 quantile and Figure 2 for 0.70 quantile.

Table 1. The floor (F), ceiling (C) and interval widths (IW) for 0.30 quantile at the level of (L) 99 % and 95 %.

L	Traditional method			Bootstrap method		
	F	C	IW	F	C	IW
99 %	1.28015	13.01153	11.73138	5.37775	12.39715	7.01940
95 %	3.04691	11.37413	8.32722	6.11508	11.41834	5.30326

Table 2. The floor (F), ceiling (C) and interval widths (IW) for 0.70 quantile at the level of (L) 99 % and 95 %.

L	Traditional method			Bootstrap method		
	F	C	IW	F	C	IW
99 %	11.37994	41.64601	30.26607	15.86223	36.97318	21.11095
95 %	14.28626	36.15094	21.86468	18.30130	34.08008	15.77878

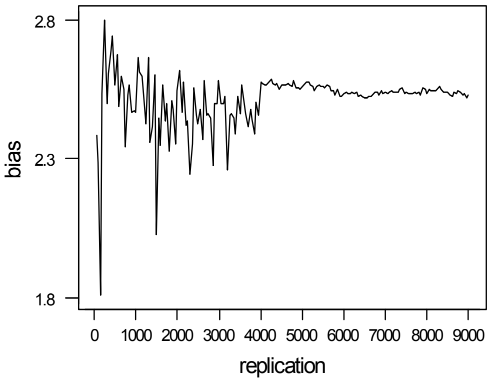

Figure 1. Plot between replication and bias 0.30 quantile.

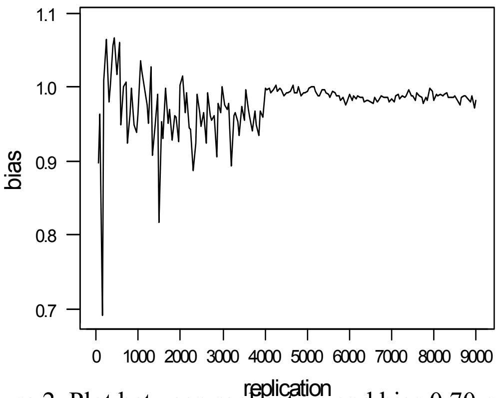

Figure 2. Plot between replication and bias 0.70 quantile.

Table 3. Comparison interval widths for 0.10 and 0.20 quantile at level of 99 % and 95 %.

Method	$t_{0.30}$		$t_{0.70}$	
	99 %	95 %	99 %	95 %
Traditional	11.73138	8.32722	30.26607	21.86468
Bootstrap percentile	7.01940	5.30326	21.11095	15.77878
Difference interval	4.71198	3.02396	9.15512	6.08590

Comparison of interval widths

Table 3 gives the interval widths of the 0.30 and 0.70 quantile on two parameters exponential distribution under multiple type-II censoring on complex case using the traditional method and the bootstrap percentile method

CONCLUSION

Bootstrap percentile method is a more potential method in constructing interval estimation for quantile on two parameters exponential distribution under multiple type-II censoring on complex case than the traditional method whereby it gives a shorter interval. And this method does not need an assumption that the sample has to have an exponential distribution.

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